ADVANCES IN MATHEMATICS 25, 191–215 (1977)

# Galois Connections and the Leray Spectral Sequence

KENNETH BACLAWSKI*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

## 1. INTRODUCTION

The intent of this paper is to present homological versions of several combinatorial theorems including Rota's theorem on the Möbius functions of ordered sets joined by a Galois connection and the Crapo complementation theorem. In so doing we strengthen and generalize these results. In addition we make a clear connection between these combinatorial theorems and various forms of the Leray spectral sequence.

The basic tool we use for presenting homology is the commutative diagram. As noted in [2], this notion is equivalent to the notion of a sheaf on a suitable topological space associated with the ordered set. In that paper we established that there were significant connections between sheaf cohomology and combinatorial notions such as the Möbius function and Whitney numbers of the first kind.

We begin in Section 2 by presenting the terminology of ordered sets and diagrams. The notation we use here is rather more simplified than that of [2]; moreover, we introduce the notion of an "augmented" diagram, which is more appropriate for the combinatorics we present. See also the presentation in [4].

In the next section we develop the concept of a Galois connection on three levels. The first is the usual notion of a Galois correspondence between ordered sets. Next we consider the analogous situation for order-preserving relations. Here we find, in contrast to the case for maps, where Galois correspondences are quite rare, that every order-preserving relation has a Galois adjoint, in fact, that it has many. Finally we consider multirelations. The lack of uniqueness of the Galois adjoint for relations now disappears, and we find that (under a suitable finiteness condition) every multirelation has a unique Galois adjoint. The significance of this adjoint appears in Section 5, where we see that it can be interpreted in terms of the Möbius function of the "fibers" of the order-preserving relation.

* This work was completed while the author was supported by an NSF Graduate Fellowship.




ISSN 0001-8708

In Section 4, we develop various forms of the Leray spectral sequence in a rather different manner than that usually found in the literature. We take a more combinatorial approach to the Leray spectral sequence. In fact, the proof we present is virtually identical to that of a theorem of Crapo [7, Theorem 1]. The results of this section could also be derived from the usual Leray spectral sequence using Artin's theory of Grothendieck topologies [1].

In the last two sections we present the main results of the paper. In Section 5 we present the homological analog of Rota's theorem [32, Theorem 1] on ordered sets joined by a Galois connection. The last section contains the homological analogs of the Crapo complementation theorem and of a related theorem, both in Crapo [7].

## 2. Ordered Sets and Diagrams

### 2.1. *Ordered Sets*

We view ordered sets as a special type of topological space. A subset $U$ of an ordered set $P$ is *open* iff $x \in U$ and $x \leqslant y$ imply $y \in U$. We also refer to such subsets as *ascending*. A closed (or *descending*) subset of $P$ with this topology is just an open subset of the *dual ordered set* $P^*$ obtained by reversing the order relation. The minimum neighborhood of a point $x \in P$ is denoted $V_x$, while the closure of $\{x\}$ is denoted $J_x$. More generally if $S \subseteq P$ we write $V_S$ for $\bigcup_{x \in S} V_x$ and $J_S$ for $\bigcup_{x \in S} J_x = \bar{S}$.

An ordered set $P$ is said to be *locally finite* if every interval $[a, b] = \{c \mid a \leqslant c \leqslant b\}$ is finite. We say $P$ is *upper* (respectively, *lower*) *finite* if for every $x \in P$, $V_x$ (respectively, $J_x$) is finite. We often have use for distinguishing the minimum element of an ordered set (if it has one). We write $P_{\hat{0}}$ for the ordered set obtained by adjoining a minimum element to $P$ *whether or not* $P$ already has such an element. So if we write $Q = P_{\hat{0}}$, then $\hat{0} \in Q$ but $\hat{0} \notin P$. We rely on the context to make it clear whether or not we intend $\hat{0}$ to be a member of a given ordered set.

### 2.2. *The Möbius Function*

The Möbius function $\mu$ (Rota [32]) of a locally finite ordered set $P$ is the unique function $\mu$: $P \times P \to \mathbb{Z}$ such that:

(a) $\mu(a, b) = 0$ if $a \nleqslant b$,

(b) $\mu(a, a) = 1$ for all $a \in P$,

(c) $\sum_{a \leqslant b \leqslant c} \mu(a, b) = 0$ for all $a < c$ in $P$.

We sometimes use $\mu_P$ for $\mu$ when some confusion may arise. In addition, to avoid overly cumbersome notation we write $\mu_P(x)$ to mean $\mu_{P_{\hat{0}}}(\hat{0}, x)$ when $x \in P$.

## 2.3. *Relations and Multirelations*

Let $P$ and $Q$ be sets. A *relation* or *correspondence from* $P$ *to* $Q$ is a subset $A \subseteq P \times Q$. We define the *image* and *inverse image* in the obvious way: for $x \in P$ and $y \in Q$,

$$A(x) = \{z \in Q \mid (x, z) \in A\},$$
$$A^{-1}(y) = \{w \in P \mid (w, y) \in A\},$$

and we extend these to sets by union, e.g., $A(S) = \bigcup_{s \in S} A(s)$. We write $A$: $P \to Q$ for relations just as one does for functions.

A (*signed*) *multiset* is a set with multiplicities (possibly negative, but necessarily integral). A *multirelation* (or *integral matrix*) $A$: $P \to Q$ is a multisubset of $P \times Q$ or equivalently an element of $\prod_{(x,y) \in P \times Q} \mathbb{Z}$. We write $A(x, y)$ for the $(x, y)$-entry of $A$, and we regard a relation as a multirelation in the obvious way. The direct and inverse images are multisets in the obvious manner, provided all summations involved are finite. Thus the multiplicity of $y$ in $A(x)$ is $A(x, y)$ and $A(S) = \sum_{s \in S} A(s)$ for $S \subseteq P$.

We would like to define a notion of order-preserving relation. Since a relation may be viewed as a function from the domain to subsets of the range, this requires that we extend the order relation to subsets. The most natural way to do this so that the usual order relation and the usual containment are special cases is as follows. Let $S, T \subseteq P$ be subsets of an ordered set. We say $S$ is *order-contained* in $T$ ($S \leqslant T$) if for all $s \in S$, there is some $t \in T$ such that $s \leqslant t$. Equivalently $S \leqslant T$ iff $J_S \subseteq J_T$.

We say that a relation $A$: $P \to Q$ is *order-preserving* if $x \leqslant y$ in $P$ implies $A(x) \leqslant A(y)$ in $Q$.

PROPOSITION 2.1. *For any relation* $A$: $P \to Q$ *of ordered sets, the following are equivalent*:

(1) *$A$ is order-preserving*,

(2) *$U \subseteq Q$ is ascending implies $A^{-1}(U)$ is ascending in $P$.*

*Proof.* (1) $\Rightarrow$ (2). Let $U \subseteq Q$ be ascending, $x \in A^{-1}(U)$ and $y \geqslant x$. By (1), $A(x) \leqslant A(y)$. Now $x \in A^{-1}(U)$ implies that there is some $z \in U$ such that $z \in A(x)$. Hence there is some $w \in A(y)$ such that $w \geqslant z$. Since $U$ is ascending, $w \in U$. Hence $y \in A^{-1}(w) \subseteq A^{-1}(U)$. So $A^{-1}(U)$ is ascending in $P$.

(2) $\Rightarrow$ (1). Let $x \leqslant y$ in $P$ and $z \in A(x)$. By (2), $A^{-1}(V_z)$ is ascending in $P$. Now $x \in A^{-1}(z) \subseteq A^{-1}(V_z)$ so $y \in A^{-1}(V_z)$. Hence there is some $w \in V_z$ such that $w \in A(y)$. Therefore $A(x) \leqslant A(y)$ and it follows that $A$ is order-preserving. Q.E.D.

We make no attempt to define a notion of order-preserving multirelation.

### 2.4. *Lattice Hemimorphisms*

Order-preserving relations are closely related to lattice maps which preserve suprema. We are primarily interested in the lattice $2^P$ of ascending subsets of an ordered set $P$. This lattice is complete and completely distributive.

We say an order-preserving map $f: L \to M$ of complete lattices is an (*upper*) *hemimorphism* if it preserves arbitrary suprema as well as $\hat{0}$.

PROPOSITION 2.2. *Let $P, Q$ be ordered sets. For an order-preserving map $f$: $2^Q \to 2^P$, the following are equivalent*:

(1) *$f$ is an upper hemimorphism*;

(2) *for some order-preserving relation $A$: $P \to Q$, $f(U) = A^{-1}(U)$ for all $U \in 2^Q$.*

*Proof.* (2) $\Rightarrow$ (1) follows from Proposition 2.1.

(1) $\Rightarrow$ (2). Define a relation $T$: $Q \to P$ by $T(z) = f(V_z)$. Let $A$: $P \to Q$ be the inverse relation to $T$. Let $U \in 2^Q$. Then $U = \bigcup_{z \in U} V_z$, so $A^{-1}(U) = \bigcup_{z \in U} A^{-1}(z) = \bigcup_{z \in U} T(z) = \bigcup_{z \in U} f(V_z) = f(\bigcup_{z \in U} V_z) = f(U)$. That $A$ is order-preserving follows from Proposition 2.1. Q.E.D.

We leave it to the reader to formulate and prove the appropriate result obtained by dropping the condition that $f$ preserve $\hat{0}$ in Proposition 2.2.

### 2.5. *Diagrams*

For simplicity in the exposition of the sequel we fix a commutative ring $R$ (with 1) once and for all. All modules to be considered are $R$-modules, and all morphisms are $R$-homomorphisms.

A (*commutative*) *diagram* (*with values in the category of R-modules*) is a collection of modules and morphisms joining these modules in such a way that all "paths" of morphisms joining a pair of objects give rise to the same morphism by composition. We say a diagram $D$ is *over an ordered set $P$* if the underlying pattern of $D$ is $P$. The module corresponding to $x \in P$ is called the *stalk $D_x$ at $x$*, and the morphism $\rho_{xy} : D_x \to D_y$ for $x \leqslant y$ is called the *structure morphism* from $x$ to $y$. A *codiagram* over $P$ is a diagram over $P^*$.

There are many ways to generate new diagrams from old ones. Any natural operation on modules automatically gives rise to a corresponding operation on diagrams. If $S \subseteq P$, then we may *restrict* a diagram $D$ to $S$; we write $D \mid S$ for the restriction. A subset $S$ of $P$ is *convex* (or *locally closed*) if $x, y \in S$ implies $[x, y] \subseteq S$. If $S$ is a convex subset of $P$, then we may also restrict to $S$ in another way. Write $D[S]$ for the diagram on $P$ such that:

(1) $$D[S]_x = D_x \quad \text{if } x \in S, \qquad = 0 \quad \text{if } x \notin S;$$

$$(2)\quad \rho_{xy} : D[S]_x \to D[S]_y \quad \text{is } \rho_{xy} : D_x \to D_y \qquad \text{if } [x, y] \subseteq S,$$
$$\text{is } \quad 0 \qquad\qquad \text{if } [x, y] \not\subseteq S.$$

The most important structure determined by a diagram $D$ is its *cohomology*. For the algebraic and categorical significance of cohomology see one of the many treatments such as Artin [1], Godement [20], or Deheuvels [10]. Our only concern here is how one computes the cohomology.

Let $D$ be a diagram on $P$. We define the *cochain complex* $C^n(D)$ of $D$ as follows.

(1) $C^n(D)$ is the module of all functions on chains $a_0 < \cdots < a_n$ of length $n$ in $P$, with values in $D_{a_n}$.

(2) $d^n$: $C^n(D) \to C^{n+1}(D)$ is given on $f \in C^n(D)$ and $a_0 < \cdots < a_{n+1}$ in $P$ by $d^n(f)(a_0 < \cdots < a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0 < \cdots < \hat{a}_i < \cdots < a_{n+1}) + (-1)^{n+1} \operatorname{res}(f(a_0 < \cdots < a_n))$.

Here res denotes the structure morphism $D_{a_n} \to D_{a_{n+1}}$ of $D$. The cohomology of $D$ is the cohomology of $C^n(D)$, namely, $H^n(P, D) = \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1})$. $H^0(P, D)$ is also written $D(P)$ and $\Gamma(D)$. In the terminology of category theory $H^n$ is the $n$th right derived functor of $\Gamma$.

A special case worth noting is that of the *constant diagram* $\tilde{M}$, where $M$ is a module. This is the diagram all of whose stalks are $M$ and all of whose structure morphisms are $\mathrm{id}_M : M \to M$. It is evident from the above cochain complex that $H^n(P, \tilde{M})$ is the ordinary simplicial cohomology $H^n(P, M)$ of $P$ with coefficients in $M$. Here $P$ is regarded as a simplicial complex in the usual way: its vertices are the elements of $P$ and its simplices are the finite chains of $P$.

Another special case is the "skyscraper" diagram $\tilde{M}[\{x\}]$, which we abbreviate to $M[x]$. A skyscraper diagram consists of a stalk $M$ at $x$, while all other stalks are zero.

Of course there are dual notions to all of the above for codiagrams. As the results for this case are so similar we generally omit special reference to them.

## 2.6. *Augmented Diagrams*

An *augmented diagram* on an ordered set $P$ is simply a diagram on $P_{\hat{0}}$. The distinction between an augmented diagram and an ordinary diagram appears only when one computes its cohomology.

Let $D$ be an augmented diagram on $P$. The *augmented cochain complex* $\tilde{C}^n(D)$ of $D$ is identical to the cochain complex $C^n(D \mid P)$ except that we have an additional term in degree $-1$.

$$(1)\quad \tilde{C}^n(D) = C^n(D \mid P), \qquad \text{for } n \geqslant 0$$
$$= D_{\hat{0}}, \qquad \text{for } n = -1.$$

(2) $\tilde{d}^n$: $\tilde{C}(D) \to \tilde{C}^{n+1}(D)$ coincides with $d^n$: $C^n(D \mid P) \to C^{n+1}(D \mid P)$ for

$n \geqslant 0$ while $\tilde{d}^{-1}$: $\tilde{C}^{-1}(D) \to \tilde{C}^0(D)$ is defined on $f \in \tilde{C}^{-1}(D) = D_{\hat{0}}$ by

$$\tilde{d}^{-1}(f)(a) = \operatorname{res}(f),$$

where res denotes the structure morphism $D_{\hat{0}} \to D_a$. In effect one may regard $\tilde{C}^{-1}(D)$ as being the module of functions defined on "empty" chains.

The cohomology of an augmented diagram $D$ is defined to be $\tilde{H}^n(P, D) = \operatorname{Ker}(\tilde{d}^n)/\operatorname{Im}(\tilde{d}^{n-1})$. Note that $\tilde{H}^n(P, D) = H^n(P, D)$ for $n \neq 0, -1$.

Most diagrams we use also have augmented versions. For example, as an augmented diagram $\tilde{M}$ is the constant diagram $\tilde{M}$ on $P_{\hat{0}}$. As a result $\tilde{H}^n(P, \tilde{M})$ coincides with $\tilde{H}^n(P, M)$, the reduced simplicial cohomology of $P$ with coefficients in $M$.

## 3. Galois Connections

Galois connections were first introduced by Ore [28], and have been studied extensively since then, for example, Everett [16], Raney [30], Derdérian [11–14]. They are defined as follows. Let $P$ and $Q$ be ordered sets. A *Galois connection* between $P$ and $Q$ is a pair of maps $\alpha$: $P \to Q$, $\beta$: $Q \to P$ satisfying

(1) $\alpha$ and $\beta$ are order reversing;

(2) for $x \in P$, $\beta(\alpha(x)) \geqslant x$ and for $y \in Q$, $\alpha(\beta(y)) \geqslant y$.

Under these conditions the maps $\beta \circ \alpha$ and $\alpha \circ \beta$ are closure relations on $P$ and $Q$, respectively, and the sets of closed elements in $P$ and in $Q$ are anti-isomorphic.

It was recognized very early in the theory of Galois connections by Everett [16] that either map of a Galois connection determines the other. Since we would like to stay in the category of order-preserving maps, we state Everett's result in this context.

Proposition 3.1 (Everett). *For an order-preserving map $\alpha$: $P \to Q$, the following are equivalent*:

(1) *for $y \in Q$, $\alpha^{-1}(V_y)$ has a minimum element in $P$*;

(2) *there is an order-preserving map $\beta$: $Q \to P$ such that the pair $(\alpha, \beta)$ is a Galois connection between $P^*$ and $Q$. The map $\beta$ is unique if it exists*; *indeed*, *it is given by any of the following equivalent conditions*:

(a) $\alpha(x) \geqslant y \Leftrightarrow x \geqslant \beta(y)$, *for* $x \in P, y \in Q$;

(b) $\alpha^{-1}(V_y) = V_{\beta(y)}$, *for* $y \in Q$;

(c) $\beta^{-1}(J_x) = J_{\alpha(x)}$, *for* $x \in P$.

Note that condition (a) above is an example of the adjointness condition in category theory. This was first noticed by Waterman [35] who developed a very general categorical context for Galois connections. The set of all Galois connections between two ordered sets is a tensor product with respect to appropriately defined categories. See Waterman [35] or Shmuely [33] for a development of the properties of this tensor product.

For obvious reasons we call an order-preserving map satisfying Proposition 3.1 (1) an *(upper) Galois map*. The dual condition gives us a *lower Galois map*. The map $\beta$ above is a lower Galois map, and we call it the *lower adjoint of* $\alpha$, while $\alpha$ is the *upper adjoint* of $\beta$. In the literature (as for example in Shmuely [33]) lower Galois maps are also referred to as "residuated" and upper Galois maps as "residual." This is an unfortunate terminology which ignores the very clear-cut relationship of such maps to Galois connections.

In the context of order-preserving relations from $P$ to $Q$, the existence of adjoints becomes much simpler. In the following we write $S \geqslant x$ to mean $S \geqslant \{x\}$ where the inequality denotes order-containment.

PROPOSITION 3.2. *Let $P$ be an ordered set. Then for any order-preserving relation $A: P \to Q$, there is an order-preserving relation $B$ from $Q^*$ to $P^*$ such that for all $(x, y) \in P \times Q$, $A(x) \geqslant y$ in $Q \Leftrightarrow B(y) \geqslant x$ in $P^*$.*

*Proof.* Define $B$ to be the relation from $Q$ to $P$ such that $B(y) = A^{-1}(V_y)$. Let $y \leqslant y'$ in $Q$ (so that $y \geqslant y'$ in $Q^*$). Then $A^{-1}(V_y) \supseteq A^{-1}(V_{y'})$ and hence $B(y) \geqslant B(y')$ in $P^*$. Therefore $B$ is order-preserving.

Now $\{x \mid A(x) \geqslant y\} = A^{-1}(V_y)$, while $\{x \mid B(y) \geqslant x \text{ in } P^*\} = V_{B(y)} = V_{A^{-1}(V_y)} = A^{-1}(V_y)$, since $A$ is order-preserving. Thus $A(x) \geqslant y$ in $Q \Leftrightarrow B(y) \geqslant x$ in $P^*$. Q.E.D.

The lower adjoint of a relation is generally not unique; however, the relation defined in the above proof enjoys many distinguishing properties. It is the largest lower adjoint of $A$, and it is *self-adjoint* in the sense that $B^{-1}(x) \leqslant y$ in $Q \Leftrightarrow B(y) \geqslant x$ in $P^*$. We summarize the properties of $B$ in the following:

PROPOSITION 3.3. *Given an order-preserving relation $A$: $P \to Q$, there is a relation $\bar{A}$ containing $A$ such that*

(1) *$\bar{A}$ is the largest relation inducing the same hemimorphism as $A$ induces from $2^Q$ to $2^P$;*

(2) *$\bar{A}$ is self-adjoint and is the largest upper adjoint of $A^{-1}$;*

(3) *$\bar{A}^{-1}$ is order-preserving from $Q^*$ to $P^*$ and $\bar{A}^{-1} = \overline{(\bar{A}^{-1})}$.*

We call $\bar{A}$ the *saturation* of $A$. In what follows it is often the case that a property of $A$ depends only on $\bar{A}$ or equivalently on the induced map from $2^Q$ to $2^P$.

*Proof.* We define the saturation $\bar{A}$ to be simply the relation "$A(x) \geqslant y$" or equivalently $\bar{A}(x) = J_{A(x)}$. Thus $\bar{A}$ contains $A$. Now $\bar{A}^{-1}(y) = \{x \mid y \in \bar{A}(x) = J_{A(x)}\} = \{x \mid y' \geqslant y$ such that $y' \in A(x)\} = A^{-1}(V_y)$. Therefore $\bar{A}$ is order-preserving and induces the same map as $A$ from $2^Q$ to $2^P$. Let $T$ be another relation with this property. Then $T^{-1}(y) \subseteq T^{-1}(V_y) = A^{-1}(V_y) = \bar{A}^{-1}(y)$. So $T \subseteq \bar{A}$. This gives (1).

Since $\bar{A}^{-1}(y) = A^{-1}(V_y)$, $\bar{A}^{-1}$ is the relation $B$ defined in the proof of Proposition 3.2. Since $B$ is clearly saturated, we get (3). It is now easy to show (2). Q.E.D.

There are numerous equivalent ways that one may express the concept of a saturated order-preserving relation. One of these is the notion of an ordered set obtained by "joining together" the domain and range of an order-preserving relation. Let $A: P \to Q$ be an order-preserving relation. The *A-join* of $P$ and $Q$, written $P +_A Q$, is the ordered set whose underlying set is the disjoint union of $P$ and $Q$ and whose ordering is given by $x \geqslant y$ if and only if one of the following hold:

(1) $x \geqslant y$ in $P$,

(2) $x \geqslant y$ in $Q$,

(3) $x \in P$, $y \in Q$, and $A(x) \geqslant y$.

Note that $P +_A Q$ is an ordered set precisely when $A$ is order-preserving. The $A$-join includes both the disjoint union $P + Q$ and the ordinal sum $Q \oplus P$ [5, p. 198] as special cases. It is easy to see that $P +_A Q$ depends only on the saturation of $A$ and that one can recover the saturation of $A$ from $P +_A Q$.

We summarize some of the many equivalent interpretations of a saturated order-preserving relation in the following:

PROPOSITION 3.4. *The following are equivalent:*

(1) *a saturated order-preserving relation* $A: P \to Q$,

(2) *an upper hemimorphism* $f: 2^Q \to 2^P$,

(3) *an order-preserving map* $g: P \to 2^{Q^*}$,

(4) *an order-preserving map* $h: Q^* \to 2^P$,

(5) *an ordering* $(P +_A Q)$ *on the disjoint union of* $P$ *and* $Q$ *such that* $P$ *is ascending and* $Q$ *is descending,*

(6) *a descending subset of* $P^* \times Q$.

The closed set in (6) is $\bar{A}$ regarded as an ordered subset of $P^* \times Q$. From (5) and (6) we get alternative ways to express a saturated order-preserving relation in terms of ordinary order-preserving maps. From (5) we have inclusions:

$$P \xrightarrow{j} P +_A Q \xleftarrow{i} Q,$$

and from (6) we have projections

$$P \xleftarrow{p} \bar{A} \xrightarrow{q} Q.$$

In either case we may recover $\bar{A}$ by composition: $\bar{A} = (i)^{-1} \circ j = q \circ (p)^{-1}$. In the latter case such a pair of maps is what is usually called a correspondence. Any relation $A$ from $P$ to $Q$ is obtainable from a correspondence, and saturation corresponds to closing $A$ in $P^* \times Q$.

When $(\alpha, \beta)$ is a Galois connection between $P^*$ and $Q$, $\beta$ is far from being unique as a relation which is lower adjoint to $\alpha$; however, $\beta$ is the smallest order-preserving relation which is lower adjoint to $\alpha$. In the case of relations there may be no smallest lower adjoint. The condition that $B$ is the lower adjoint of $A: P \to Q$ may be written: $A^{-1}(V_y) = V_{B(y)}$ for all $y \in Q$. Consider $P$, for example, to be the integers with the usual order, $Q$ to be the singleton $\{y\}$, and $A: P \to Q$ the map such that $A(x) = y$ for all $x$. Then $V_{B(y)}$ must be all of $P$. There is no smallest set $B(y)$ with this property. We return to this example later.

If we assume $P$ is lower finite, then there is a smallest order-preserving relation $B$ which is lower adjoint to $A$, namely, let $B(y)$ be the minimal elements of $A^{-1}(V_y)$ for all $y \in Q$.

So far we have considered adjoints of order-preserving maps and order-preserving relations. We now consider the case of multirelations. This means that we are now working in the context of multisets. The adjointness condition must now be satisfied with multiplicities taken into account as well.

Definition. Let $A: P \to Q$, $B: Q^* \to P^*$ be multirelations. We say $A$ is the *upper adjoint* of $B$ (and $B$ the *lower adjoint* of $A$) iff $B^{-1}(J_x) = J_{A(x)}$ as multisets for all $x \in P$. Equivalently, we could have required that $A^{-1}(V_y) = V_{B(y)}$ for $y \in Q$ by the next result.

Lemma. *The following are equivalent*:

(1) $B^{-1}(J_x) = J_{A(x)}$ *as multisets for all* $x \in P$;

(2) $A^{-1}(V_y) = V_{B(y)}$ *as multisets for all* $y \in Q$;

(3) $\sum_{x' \leqslant x} B(y, x') = \sum_{y' \geqslant y} A(x, y')$ *for all* $(x, y) \in P \times Q$.

*Proof.* Let $x \in P$, $y \in Q$. Then $\mathrm{mult}_y(B^{-1}(J_x)) = \sum_{x' \leqslant x} \mathrm{mult}_y B^{-1}(x') = \sum_{x' \leqslant x} B(y, x')$ and $\mathrm{mult}_y(J_{A(x)}) = \sum_{y' : y \in J_{y'}} \mathrm{mult}_{y'} A(x) = \sum_{y' \geqslant y} A(x, y')$. So (1) is equivalent to (3). Similarly (2) is equivalent to (3). Q.E.D.

We may now ask whether there are adjoints to multirelations. An obvious necessary condition for $A: P \to Q$ to have a lower adjoint is that $A^{-1}(V_y)$ be a well-defined multiset. The example above of an order-preserving map without a smallest lower adjoint is also an example of a multirelation without a lower adjoint. Thus this condition is not sufficient. Hower, if we demand that $P$ be lower finite, we not only have adjoints but they are also uniquely determined.

THEOREM 3.5. *Let $A: P \to Q$ be a multirelation such that $A^{-1}(V_y)$ is a multiset for all $y \in Q$. If $P$ is lower finite, then $A$ has a unique lower adjoint $B$. If $A$ is an upper Galois map then $B$ is its lower adjoint as an order-preserving map.*

*Proof.* By the assumption on $A$, $\sum_{y' \geqslant y} A(x, y')$ is finite for all $x \in P$, $y \in Q$. Fix $y \in Q$. Then $B(y, x)$ is determined uniquely by the values of $B(y, x')$ for $x' < x$ by the formula

$$B(y, x) = \sum_{y' \geqslant y} A(x, y') - \sum_{x' < x} B(y, x').$$

Since $P$ is lower finite, $B$ exists and is unique by induction.

Suppose that $\alpha$ is an upper Galois map with lower adjoint $\beta$ as an order-preserving map. Then $\sum_{y' \geqslant y} \alpha(x, y')$ is 1 or 0 depending on whether or not $\alpha(x) \geqslant y$, respectively. Similarly for $\sum_{x' \leqslant x} \beta(y, x')$ and $x \geqslant \beta(y)$. Since $\alpha(x) \geqslant y \Leftrightarrow x \geqslant \beta(y)$, $\beta$ is also the lower adjoint of $\alpha$ as a multirelation. Q.E.D.

When $A$ is an order-preserving map, the lower adjoint as determined above can be quite complicated. See Theorem 5.5 for an explicit formula. The lower adjoint is a relation only in the following case:

PROPOSITION 3.6. *The lower adjoint in the multiset sense of an order-preserving map $f: P \to Q$, where $P$ is lower finite, will be a relation if and only if $f^{-1}(V_y)$ is a disjoint union of principal filters (possibly empty) for all $y \in Q$. When this is the case the lower adjoint is the smallest lower adjoint to $f$ in the sense of relations.*

*Proof.* Let $B$ be the lower adjoint of $f$. By the inductive definition of $B$ in Theorem 3.5, $B(y, x) = 1$ if $x$ is a minimal element of $f^{-1}(V_y)$. If there is no element of $P$ lying above two or more minimal elements of $f^{-1}(V_y)$, then $B(y)$ is the set of minimal elements of $f^{-1}(V_y)$. On the other hand, if there are such elements, we may choose a minimal such element $x$. If $x$ lies above exactly $n \geqslant 2$ minimal elements of $f^{-1}(V_y)$, then one easily computes that $B(y, x) = 1 - n$ so that $B$ cannot be a relation in this case. Q.E.D.

We now show that with respect to Möbius inversion, the lower adjoint in the above multiset sense is precisely the right context for Rota's "Main Theorem" [32].

THEOREM 3.7. *Let $f: P \to Q$ be an order-preserving map of lower finite* *ordered sets. Let B be the lower adjoint of f in the multiset sense. Then for every $x \in P$,*

$$\mu_P(x) = \sum_{y \in B^{-1}(x)} \mu_Q(y),$$

*where the sum is in the sense of multisets.*

*Proof.* By the definition of $\mu_P$ we need only show that for every $x \in P$, $\sum_{x' \leqslant x} \sum_{y \in B^{-1}(x')} \mu_Q(y) = -1$. Now

$$\sum_{x' \leqslant x} \sum_{y \in B^{-1}(x')} \mu_Q(y) = \sum_{x' \leqslant x} \sum_{y \in Q} B(y, x') \mu_Q(y) = \sum_{y \in Q} \sum_{x' \leqslant x} B(y, x') \mu_Q(y)$$

$$= \sum_{y \in Q} \sum_{y' \geqslant y} f(x, y') \mu_Q(y) = \sum_{y \in J_{f(x)}} \mu_Q(y) = -1. \text{ Q.E.D.}$$

The above theorem does not extend to relations. The crucial property of $f$ needed for the theorem is that $J_{f(x)}$ is homologically trivial. In the sequel we see more clearly why this requirement is important and what results when we try to relax it.

A coloring of a simplicial complex is an example of an order-preserving map that satisfies the condition of Proposition 3.6. Here we use the interpretation of a coloring in the geometric theory of coloring of Fisk [18]. A 4-*coloring* of a pure two-dimensional simplicial complex $\Sigma$ is a rank-preserving, nondegenerate order-preserving map $f: \Sigma \to \partial\Delta^3$, where $\partial\Delta^3$ is the boundary of the tetrahedron. It is easy to see that $f$ is not Galois in general, but its lower adjoint in the sense of multirelations is a relation.

## 4. The Leray Spectral Sequence

Let $P, Q$ be ordered sets and $f: P \to Q$ an order-preserving map. Let $D$ be a diagram on $P$. By the standard constructions of diagrams, we can "push-out" $D$ to a diagram on $Q$. We call this the *direct image* $f_*D$. Godement [20] and Artin [1] given more thorough descriptions of these operations. For the sequel we need only the definition:

$$(f_*D)_y = D(f^{-1}(V_y)) = H^0(f^{-1}(V_y), D), \qquad y \in Q.$$

The structure morphisms of $f_*D$ are induced by the inclusion $f^{-1}(V_z) \subseteq f^{-1}(V_y)$ whenever $y \leqslant z$.

The definition of the direct image does not require that $f$ be order-preserving or even that $f$ be a function. When $A$ is a relation from $P$ to $Q$, the direct image is defined by

$$(A_*D)_y = D(A^{-1}(V_y)) = H^0(A^{-1}(V_y), D), \qquad y \in Q.$$

In the obvious way we may generalize $A_*D$ by using $H^p$ instead of $H^0$. We call these the *higher direct images*:

$$(R^pA_*D)_y = H^p(A^{-1}(V_y), D).$$

The key property of the higher direct images is that for any short exact sequence of diagrams on $P$

$$0 \to D \to E \to F \to 0,$$

we have a natural long exact sequence of diagrams on $Q$

$$0 \to A_*D \to A_*E \to A_*F \to R^1A_*D \to R^1A_*E \to R^1A_*F \to R^2A_*D \to \cdots.$$

To see that the above sequence is exact one need only notice that the stalks on each $y \in Q$ form a long exact sequence in the usual sense. Since the usual long exact sequence is natural, the above sequence of diagrams is a natural sequence of diagrams. In the language of category theory, the $R^pA_*$ are the right derived functors of $A_*$ . Note that the letter "$R$" here denotes "right" and has nothing to do with the ring $R$. No confusion should result since the two uses of $R$ occur in different contexts.

The direct image and higher direct images also have augmented versions with analogous properties to the usual direct images. Let $A\colon P \to Q$ be a relation, and $D$ a diagram on $P$. The *augmented higher direct image* $R^pA_*D$ is the diagram on $Q_{\hat{0}}$ such that

$$\begin{aligned}(R^pA_*D)_y &= H^p(A^{-1}(V_y), D), &&\text{if } y \in Q,\\ &= H^p(P, D), &&\text{if } y = \hat{0}.\end{aligned}$$

The structure morphisms of $R^pA_*D$ are induced by the inclusions $A^{-1}(V_z) \subseteq A^{-1}(V_y)$ for $y \leqslant z$ and $P \supseteq A^{-1}(V_y)$ for $\hat{0} < y$. We distinguish the augmented direct images from the usual ones by using $\tilde{H}$ instead of $H$ when taking their cohomology.

We could also allow $D$ to be an augmented diagram. By using augmented cohomology, $\tilde{H}$, instead of ordinary cohomology, $H$, we could get the augmented higher direct images $\tilde{R}^pA_*D$. We won't use these explicitly in the sequel.

THEOREM 4.1. *Let $A\colon P \to Q$ be an order-preserving relation, and let $D$ be a diagram on $P$. Then there is a spectral sequence*

$$E_2^{p,q} = \tilde{H}^p(Q, R^qA_*D) \Rightarrow H^{n+1}(P +_A Q, D),$$

*where we regard $D$ as a diagram on $P +_A Q$ by extending it by zero.*

*Proof.* The cochain complex $C^n = C^n(D)$ for computing the cohomology of $D$ on $P +_A Q$ is as follows:

(1) $C^n$ is the module of functions on chains $a_0 < \cdots < a_n$ in $P +_A Q$ such that $a_n \in P$, with values in $D_{a_n}$.

(2) $d\colon C^n \to C^{n+1}$ for $f \in C^n$ and $a_0 < \cdots < a_{n+1}$, a chain in $P +_A Q$ such that $a_{n+1} \in P$ is given by:

$$d(f)(a_0 < \cdots < a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0 < \cdots < \hat{a}_i < \cdots < a_{n+1}) \quad \text{if } a_n \notin P,$$

$$= \sum_{i=0}^{n} (-1)^i f(a_0 < \cdots < \hat{a}_i < \cdots < a_{n+1})$$

$$+ (-1)^{n+1} \operatorname{res} f(a_0 < \cdots < a_n) \quad \text{if } a_n \in P.$$

Here res is the structure morphism $D_{a_n} \to D_{a_{n+1}}$ of the diagram $D$.

Each of the $C^n$ can be decomposed according to the number of elements in the chain $a_0 < \cdots < a_n$ which belong to $P$. Thus define

(1) $N^{p,q}$ to be the module of functions on chains $a_0 < \cdots < a_p < b_0 < \cdots < b_q$ in $P +_A Q$ such that the $a_i \in Q$ and the $b_j \in P$. We allow $p = -1$, in which case $a_0 < \cdots < a_p$ is the empty chain of $Q$. We do not allow $q = -1$.

(2) $d'\colon N^{p,q} \to N^{p+1,q}$ is given on $f \in N^{p,q}$ by

$$d'(f)(a, b) = \sum_{i=0}^{p+1} (-1)^i f(a_0 < \cdots < \hat{a}_i < \cdots < a_{p+1} < b_0 < \cdots < b_q).$$

(3) $d''\colon N^{p,q} \to N^{p,q+1}$ is given on $f \in N^{p,q}$ by

$$d''(f)(a, b) = \sum_{i=0}^{q} (-1)^{p+i+1} f(a_0 < \cdots < a_p < b_0 < \cdots < \hat{b}_i < \cdots < b_{q+1})$$

$$+ (-1)^{p+q+2} \operatorname{res} f(a_0 < \cdots < a_p < b_0 < \cdots < b_q).$$

Clearly $C^{n+1} \cong \bigoplus_{p+q=n} N^{p,q}$ and $d = d' + d''$, for $n \geqslant -1$. Therefore $\{N^{p,q}; d', d''\}$ is a double complex whose total complex satisfies $N^n = C^{n+1}$ for $n \geqslant -1$.

The computation of the abutment is easy since it is simply the cohomology of the complex $C^n$ but with the degree shifted by one.

We now compute the $E_1$-term. This is the cohomology of $N^{p,q}$ with respect to $d''$. For a fixed choice of $p \geqslant 0$ and chain $a_0 < \cdots < a_p$ in $Q$, this is just the cohomology of $D$ restricted to that part of $P$ which lies above $a_p$: namely, $A^{-1}(V_{a_p})$. When $p = -1$, there is no longer any restriction on the $b_j$'s, so we get $H^q(P, D)$. Thus the $E_1$-term is given by

(1) $E_1^{p,q} = H^q(N^{p,q}, d'')$ is the module of functions on chains $a_0 < \cdots < a_p$

of $Q$ with values in $H^q(A^{-1}(V_{a_p}), D)$ when $p \neq -1$ and with values in $H^q(P, D)$ when $p = -1$.

(2) $d_1^{p,q}$: $E_1^{p,q} \to E_1^{p+1,q}$ is the restriction of $d'$; so if $f \in E_1^{p,q}$ and $a_0 < \cdots < a_{p+1}$ is a chain of $Q$, then

$$d_1^{p,q}(f)(a_0 < \cdots < a_{p+1}) = \sum_{i=0}^{p} (-1)^i f(a_0 < \cdots < \hat{a}_i < \cdots < a_{p+1})$$
$$+ (-1)^{p+1} \operatorname{res} f(a_0 < \cdots < a_p),$$

where res: $H^q(A^{-1}(V_{a_p}), D) \to H^q(A^{-1}(V_{a_{p+1}}), D)$ is induced by the inclusion $A^{-1}(V_{a_{p+1}}) \subseteq A^{-1}(V_{a_p})$ when $p \neq -1$, and similarly when $p = -1$.

We now observe that, for fixed $q$, $d_1^{p,q}$ is the differential of the cochain complex for the augmented diagram $R^qA_*D$ on $Q$. Hence the $E_2$-term of the spectral sequence is $E_2^{p,q} = \tilde{H}^p(Q, R^qA_*D)$. Q.E.D.

We leave it as an exercise to reformulate Theorem 4.1 using an augmented diagram $D$. The next result is the "unaugmented" case of Theorem 4.1.

COROLLARY 4.2. *Let $A$: $P \to Q$ be an order-preserving relation, and let $D$ be a diagram on $P$. Then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(Q, R^qA_*D) \Rightarrow H^n$$

*whose abutment appears in a natural long exact sequence* $0 \to H^0(P +_A Q, D) \to H^0(P, D) \to H^0 \to H^1(P +_A Q, D) \to H^1(P, D) \to H^1 \to \cdots$.

By "first quadrant" we mean simply that $E_2^{p,q} = 0$ unless $p \geqslant 0$, $q \geqslant 0$.

*Proof.* Let

$$\begin{aligned} \bar{N}^{p,q} &= N^{p,q} \qquad &&\text{if } p \neq -1 \\ &= 0 &&\text{if } p = -1. \end{aligned}$$

Then $\bar{N}$ is a subdouble complex of $N$ in the sense that the inclusions $\bar{N}^{p,q} \to N^{p,q}$ give rise to a commutative diagram. In much the same way as in the proof of Theorem 4.1, we see that $\bar{N}^{p,q}$ gives rise to the spectral sequence of this corollary.

The abutment remains to be computed. Taking the total complexes of $\bar{N}$ and $N$ we see that there is a short exact sequence of complexes: $0 \to \bar{N}^n \to N^n \to N^{-1,n+1} \to 0$. Now it is easy to see that $N^{-1,n+1} = C^{n+1}(D)$, where $C^n(D)$ is the cochain complex for the cohomology of $D$ on $P$. Since $N^n = C^{n+1}$, where $C^n$ is the cochain complex for $D$ on $P +_A Q$, the result follows by an application of the snake lemma (Cartan–Eilenberg [6, IV.3]). Q.E.D.

We can get another spectral sequence with the same abutment from the double complex $N^{p,q}$ by interchanging the roles of $p$ and $q$. Thus let $M^{p,q} = N^{q,p}$, and similarly $\bar{M}^{p,q} = \bar{N}^{q,p}$.

We compute the $E_1$-term of $M$. Here we fix $q$ and a chain $b_0 < \cdots < b_q$ in $P$ and compute the cohomology of $N^{p,q}$ with respect to $d'$. The result is easily seen to be the ordinary reduced simplicial cohomology of $J_{A(b_0)}$ with coefficients in the module $D_{b_q}$. The computation for $\overline{M}$ is the same except that we use unreduced cohomology. Therefore,

(1) $E_1^{p,q}(M) = H^q(N^{q,p}, d')$ is the module of functions on chains $b_0 < \cdots < b_p$ in $P$ with values in $\tilde{H}^q(J_{A(b_0)}, D_{b_p})$.

(2) $d_1^{p,q}$ is given on $f \in H^q(N^{q,p}, d')$ by

$$\begin{aligned} d_1^{p,q}(f)(b_0 < \cdots < b_{p+1}) = (-1)^{q+1} \operatorname{res} f(b_1 < \cdots < b_{p+1}) \\ + \sum_{i=1}^{p} (-1)^{q+i+1} f(b_0 < \cdots < \hat{b}_i < \cdots < b_{p+1}) \\ + (-1)^{p+q+2} \operatorname{res} f(b_0 < \cdots < b_p), \end{aligned}$$

where the first res is the map $\tilde{H}^q(J_{A(b_1)}, D_{b_p}) \to \tilde{H}^q(J_{A(b_0)}, D_{b_p})$ induced by $J_{A(b_0)} \subseteq J_{A(b_1)}$, while the second is the map $\tilde{H}^q(J_{A(b_0)}, D_{b_p}) \to \tilde{H}^q(J_{A(b_0)}, D_{b_{p+1}})$ induced by the structure morphism $D_{b_p} \to D_{b_{p+1}}$ of $D$.

In general $d_1^{p,q}$ does not induce a differential arising from the cochain complex of some diagram. However, there are two special cases for which it does. In the first case the first res map above reduces to the identity. In the second case the second one does. We consider each of these cases in turn and then go on to consider a few special cases of the former.

Suppose that $A$: $P \to Q$ is an order-preserving relation. For $x \in P$, we call $J_{A(x)} \subseteq Q$, the *cofiber of $A$ under $x$*. We say that *$A$ has uniform cofibers* (*over $R$*) if for $x \leqslant x'$ in $P$, the inclusion $J_{A(x)} \to J_{A(x')}$ induces an isomorphism in homology with coefficients in $R$. When this is the case we write simply $J_A$ for an arbitrary cofiber of $A$. Note that if $A$ is an order-preserving map, then $A$ clearly has uniform cofibers for then $J_{A(x)}$ is an ordered set with a maximum element $A(x)$. By the discussion above we have the following result.

PROPOSITION 4.3. *Let $A$: $P \to Q$ be an order-preserving relation with uniform cofibers, and let $D$ be a diagram on $P$. Then there is a first quadrant spectral sequence with*

$$E_2^{p,q} = H^p(P, \mathscr{H}^q(J_A, D)) \Rightarrow H^n$$

*whose abutment is the same as that in Corollary 4.2 and where $\mathscr{H}^q(J_A, D)$ is the diagram on $P$ whose stalk at $x$ is $H^q(J_A, D_x)$ and whose structure morphisms are induced by those of $D$.*

In the special case of an order-preserving map $f$: $P \to Q$, the above spectral sequence has $E_2^{p,0} = H^p(P, D)$ and all other terms vanish. Therefore, in this case, the abutment coincides with the $E_2$-term and we get the usual Leray spectral sequence:

COROLLARY 4.4 (Leray). *Let $f: P \to Q$ be an order-preserving map, and $D$ a diagram on $P$. Then there is a first quadrant spectral sequence with*

$$E_2^{p,q} = H^p(Q, R^q f_* D) \Rightarrow H^n(P, D).$$

Of course, the same result holds for any order-preserving relation having cofibers all of whose reduced homology over $R$ is trivial.

The next case we consider is that of a constant diagram. When this is the case we need no further condition on the relation $A$ in order to compute the $E_2$-term of $M^{p,q}$.

PROPOSITION 4.5. *Let $A: P \to Q$ be an order-preserving relation and let $K$ be an $R$-module. Then there is a first quadrant spectral sequence with*

$$E_2^{p,q} = H^p(P^*, R^q(\bar{A}^{-1})_* \tilde{K}) \Rightarrow H^n$$

*whose abutment is the same as that in Corollary* 4.2.

Here we regard $\bar{A}^{-1}$ as a relation from $Q^*$ to $P^*$ as in Proposition 3.3.

It seems reasonable to write $\bar{A}^*$ for $(\bar{A}^{-1})_*$ and this essentially coincides with standard terminology when $A$ is a map. We thus conclude from Corollary 4.2 and Proposition 4.5 that

$$H^p(Q, R^q A_* \tilde{K}) \Rightarrow H^n \Leftarrow H^p(P^*, R^q \bar{A}^* \tilde{K})$$

are spectral sequences converging to the same abutment.

*Proof of Proposition* 4.5. From the computation of $E_1^{p,q}(M)$ above with $D = \tilde{K}$ we note that the second res map is now the identity. Up to a sign change (which does not affect cohomology), $d_1^{p,q}$ is the differential of the cochain complex of the diagram on $P^*$ whose stalk at $x$ is $H^q(J_{A(x)}, K)$ and whose structure morphism for $x \leqslant x'$ is induced by the inclusion $J_{A(x)} \subseteq J_{A(x')}$. This gives a codiagram on $P$ and hence a diagram on $P^*$.

Now $B = \bar{A}^{-1}: Q^* \to P^*$ satisfies $B^{-1}(V_x) = \bar{A}(J_x) = J_{A(x)}$, (as in the proof of Proposition 3.3) where the first $V_x$ is the principal filter of $x$ in $P^*$ whereas the remaining terms are to be taken in $P$. Therefore $(R^q B_* \tilde{K})_x = H^q(J_{A(x)}, \tilde{K})$. Q.E.D.

The last case we consider is that for which $A$ is an upper Galois map.

PROPOSITION 4.6. *Let $f: P \to Q$ be an upper Galois map of ordered sets. Then for every diagram $D$ on $P$, $R^p f_* D = 0$ for $p > 0$.*

*Proof.* We need only the fact that the cohomology of a diagram on an ordered set with a minimum element is trivial: all higher cohomology vanishes. Thus for a diagram $D$ on $P$, $p > 0$, and $y \in Q$, we have

$$(R^p A_* D)_y = H^p(A^{-1}(V_y), D) = 0,$$

since $A^{-1}(V_y)$ has a minimum element. Q.E.D.

COROLLARY 4.7. *Let $A: P \to Q$ be a Galois map. Then for all $n \geqslant 0$*

$$H^n(P, D) \cong H^n(Q, A_* D).$$

Of course the result of Proposition 4.6 holds whenever $A^{-1}(V_y)$ is cohomologically trivial with respect to $D$ for all $y \in Q$. For example such will be the case if $A^{-1}(V_y)$ is always a disjoint union of subsets of $P$ of the form $V_x$.

For another example let $S \subseteq P$ be a subset such that for all $x \in P$, $V_x \cap S$ has trivial (reduced) simplicial cohomology (over $R$), in particular that $V_x \cap S \neq \varnothing$. Then the inclusion of $S$ into $P$ has no higher direct images with respect to the constant diagram $\tilde{R}$. Moreover, the direct image of $\tilde{R}$ over $S$ is $\tilde{R}$ over $P$. So in this case we get an isomorphism of simplicial cohomology

$$H^n(S, R) \cong H^n(P, R)$$

for all $n \geqslant 0$. This is a well-known cofinality result in simplicial topology.

We use this cofinality result in the following special case. Let $L$ be a finite lattice. The elements of $L$ which cover the minimum $\hat{0}$ of $L$ are called the *atoms* of $L$. Let $T \subseteq L$ be the set of all suprema of atoms of $L$ including $\hat{0}$ (the empty supremum). Then $S = T \backslash \{\hat{0}, \hat{1}\}$ is (dual) cofinal in $P = L \backslash \{\hat{0}, \hat{1}\}$ in the above sense, for $J_x \cap S$ has the maximum element $\sup\{a \leqslant x \mid a \text{ is an atom of } L\}$ for all $x \in P$.

## 5. COMBINATORIAL CONSEQUENCES

We derive combinatorial results by applying the Euler characteristic to both "sides" of the spectral sequences in Section 4. For this to make sense we assume henceforth that the ring $R$ is a principal ideal domain. In this case if $M = \bigoplus_{n=-\infty}^{\infty} M^n$ is a graded $R$-module of finite rank, we define the *Euler characteristic* of $M$ to be $\chi(M) = \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{rank}_R(M^n)$. For a finite doubly graded $R$-module, $M = \bigoplus_{p,q} M^{p,q}$, we define $\chi(M)$ to be $\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (-1)^{p+q} \times \operatorname{rank}_R(M^{p,q})$.

Now suppose that $D$ is a diagram on an ordered set $P$. The *closed support* of $D$ is $\{x \in P \mid D_{x'} \neq 0 \text{ for some } x' \geqslant x\} = J_{\{x \in P \mid D_x \neq 0\}}$. We say $D$ is a *finite diagram* if its closed support is finite and all its stalks are modules of finite rank over $R$. Now for the purposes of computation of cohomology we may assume $D$ is a diagram on its closed support. Evidently, if $D$ is a finite diagram, then $\bigoplus_{n=0}^{\infty} H^n(P, D)$ has finite rank. Therefore $D$ has an Euler characteristic: $\chi(D) = \sum_{n=0}^{\infty}(-1)^n \operatorname{rank}_R H^n(P, D)$.

More generally, let $A\colon P \to Q$ be an order-preserving relation with finite cofibers, and let $D$ be a finite diagram on $P$. If $S$ is the closed support of $D$, then since $A$ has finite cofibers, $J_{A(S)}$ is finite. Now the closed supports of the $R^pA_*D$ are all contained in $J_{A(S)}$. Therefore each of the $R^pA_*D$ is a finite diagram. Moreover, since $H^p$ vanishes for $p$ sufficiently large on a finite ordered set, $R^pA_*D = 0$ for $p$ sufficiently large. Therefore $\bigoplus_{n=0}^{\infty} R^pA_*D$ is a finite diagram. We conclude that as long as we restrict attention to finite diagrams and relations with finite cofibers, the Euler characteristic is a legitimate operation.

The basic result which ties the Euler characteristic to the cohomology of diagrams is the following homological version of a theorem of Rota [31, Theorem 3, Corollary 2].

LEMMA 5.1. *Let $M$ be a finite $R$-module and $P$ a lower finite ordered set. Then for all $x \in P$,*

$$\chi(M[x]) = -\operatorname{rank}_R(M) \cdot \mu_P(x).$$

*Proof.* We begin with [2, Corollary 3.1], but in homology rather than cohomology. This is the codiagram case of Lemma 5.1 with $R = M = \mathbb{Z}$. By the universal coefficient theorem we have short exact sequences:

$$0 \to H_n(\mathbb{Z}[x]) \otimes_{\mathbb{Z}} R \to H_n(R[x]) \to \operatorname{Tor}_1(H_{n-1}(\mathbb{Z}[x]), R) \to 0.$$

Now $\operatorname{Tor}_1(G, R)$, for an Abelian group $G$, vanishes if $G$ is free and has rank zero over $R$ if $G$ is finite. See Cartan–Eilenberg [6, Chap. VII, Corollary 1.7]. Thus, in general, $\operatorname{rank} H_n(\mathbb{Z}[x]) = \operatorname{rank}_R H_n(R[x])$. Applying the universal coefficient theorem once more (but over $R$) gives the lemma. Q.E.D.

PROPOSITION 5.2. *Let $P$ be an ordered set and $D$ a finite diagram on $P$. Then $\chi(D) = \sum_{x \in P} \chi(D[x])$.*

*Proof.* Restricting $D$ to its closed support, we may assume $P$ is finite. By Szpilrajn's theorem [34], there is an injective order-preserving map $f\colon P \to \mathbb{N}$. By [2, Theorem 4.1], we have a spectral sequence $E_2^{p,q} = H^{p+q}(P, D[f^{-1}(p)]) \Rightarrow H^n(P, D)$.

Now spectral sequences preserve Euler characteristics. On the left side the Euler characteristic is

$$\sum_{p,q} (-1)^{p+q} \operatorname{rank}_R H^{p+q}(P, D[f^{-1}(p)])$$

$$= \sum_{x \in P} \sum_{q} (-1)^{f(x)+q} \operatorname{rank}_R H_{f(x)+q}(P, D[x])$$

$$= \sum_{x \in P} \chi(D[x]).$$

The Euler characteristic of the abutment is $\chi(D)$. Q.E.D.

By the above proposition, all Euler characteristic computations for diagrams reduce to the case of a "skyscraper" diagram $R[x]$. For this special case Corollary 4.5 takes this form:

THEOREM 5.3. *Let $f\colon P \to Q$ be an upper Galois map of lower finite ordered sets. For $x \in P$ write $S_x = \{y \in Q \mid V_x = f^{-1}(V_y)\}$. Then for $x \in P$,*

(1) $H^n(P, R[x]) \simeq H^n(Q, \tilde{R}[S_x])$ $(n \geqslant 0)$,

(2) $\mu_P(x) = \sum_{y \in S_x} \mu_Q(y)$.

*Proof.* For (1) we simply note that $\tilde{R}[S_x] = f_* R[x]$ and then apply Corollary 4.5. For (2) we take the Euler characteristics of both sides of (1) using Lemma 5.1 and Proposition 5.2. Q.E.D.

The above theorem can be given an alternative expression in terms of a long exact sequence as follows. The set $S_x$ is the difference of the descending subsets $T_x = \{y \in Q \mid V_x \subseteq f^{-1}(V_y)\}$ and $U_x = \{y \in Q \mid V_x \subsetneqq f^{-1}(V_y)\}$. We therefore have a short exact sequence of diagrams on $Q$:

$$0 \to \tilde{R}[S_x] \to \tilde{R}[T_x] \to \tilde{R}[U_x] \to 0.$$

Since $U_x$ and $T_x$ are descending subsets of $Q$, the cohomology of $\tilde{R}[U_x]$, for example, is just the simplicial cohomology of $U_x$ with coefficients in $R$. Therefore we get

COROLLARY 5.4 (Griffiths [23, Theorem 3.2]). *Let $f\colon P \to Q$ be an upper Galois map, and let $x \in P$. With $U_x$ and $T_x$ as above, and with $K_x = \{x' \in P \mid x' < x\}$, there is a long exact sequence*:

$$0 \to \tilde{H}^{-1}(K_x, R) \to H^0(T_x, R) \to H^0(U_x, R) \to \tilde{H}^0(K_x, R) \to H^1(T_x, R) \to \cdots$$

*Proof.* If we write out the long exact sequence of the above short exact sequence and use Theorem 5.3, the result reduces to showing that $H^i(P, R[x]) \simeq$

$\tilde{H}^{i-1}(K_x, R)$. But this last result follows from the special case $f = id$: $P \to P$. Q.E.D.

Corollary 5.4 could also be stated in homological terms. In this formulation it is easy to see that it is equivalent to the result of Griffiths cited above. Other results of Griffiths [22, 23] can also be given spectral sequence proofs and interpretations.

Let $(f, g)$ be a Galois connection between $P^*$ and $Q$. Then $f$: $P \to Q$ is Galois and by Theorem 5.3(2) we have that

$$\mu_P(x) = \sum_{y \in S_x} \mu_Q(y).$$

The definition of $S_x$ immediately implies that $S_x = g^{-1}(x)$, by Proposition 3.1. In this form it is clear that Theorem 5.3 contains [32, Theorem 1]. Moreover, part (1) does not require that $R$ be a principal ideal domain nor that the ordered sets be lower finite.

In the general case we have

THEOREM 5.5. *Let $f$: $P \to Q$ be an order preserving map of lower finite ordered sets. For $x \in P$,*

$$\mu_P(x) = - \sum_{y \leqslant f(x)} \mu_{f^{-1}(V_y)}(x)\, \mu_Q(y).$$

*Proof.* We apply Theorem 4.2 to the diagram $R[x]$.

$$\begin{aligned}
\mu_P(x) &= -\chi(R[x]) \qquad \text{(Lemma 5.1)} \\
&= -\sum_{p,q} (-1)^{p+q} \operatorname{rank}_R H^p(Q, R^q f_* R[x]) \qquad \text{(Theorem 4.2)} \\
&= -\sum_q (-1)^q \chi_Q(R^q f_* R[x]) \\
&= -\sum_q \sum_{y \in Q} (-1)^q \chi_Q((R^q f_* R[x])[y]) \qquad \text{(Proposition 5.2)} \\
&= -\sum_q \sum_{y \in Q} (-1)^q \chi_Q(H^q(f^{-1}(V_y), R[x])[y]) \\
&= -\sum_q \sum_{y \in Q} (-1)^q \operatorname{rank}_R(H^q(f^{-1}(V_y), R[x]))(-\mu_Q(y)) \qquad \text{(Lemma 5.1)} \\
&= \sum_{y \in Q} \chi_{f^{-1}(V_y)}(R[x] \mid f^{-1}(V_y))\, \mu_Q(y) \\
&= \sum_{y \leqslant f(x)} -\mu_{f^{-1}(V_y)}(x)\, \mu_Q(y) \qquad \text{(Lemma 5.1).}
\end{aligned}$$

Q.E.D.

The algorithm for computing $-\mu_{f^{-1}(V_y)}(x)$ is quite straightforward. Regard it as being zero if $x \notin f^{-1}(V_y)$. Then $-\mu_{f^{-1}(V_y)}(x) = 1$ for all minimal elements $x$ of $f^{-1}(V_y)$, and for nonminimal elements $x$ of $f^{-1}(V_y)$, the formula

$$-\mu_{f^{-1}(V_y)}(x) = 1 + \sum_{x'<x} \mu_{f^{-1}(V_y)}(x)$$

determines the left-hand side inductively. Looking back at the proof of Theorem 3.5, we find that $-\mu_{f^{-1}(V_y)}(x)$ is precisely $\operatorname{mult}_y B^{-1}(x) = B(y, x)$, where $B$ is the lower adjoint of $f$ in the sense of multirelations. We conclude that the Leray spectral sequence is the homological version of Theorem 3.5, which in turn is the multiset generalization of Rota's theorem on Galois connections.

Theorems 3.5 and 5.5 are usually too complicated to apply in general. They are useful only when one has good combinatorial information about the fibers $f^{-1}(V_y)$ of the map $f$. See Baclawski [4] for an example of this. Nonetheless, one can use Theorem 5.5 to generate special cases such as the following.

PROPOSITION 5.6 (Fiber Inclusion-Exclusion). *Let $f\colon P \to Q$ be an order-preserving map of finite ordered sets. Assume that $P_{\hat{0}}$ is a lattice. Let $B\colon Q^* \to P^*$ be the smallest relation that is lower adjoint to $f$. Then for $x \in P$:*

$$\mu_P(x) = \sum_{y \leqslant f(x)} \sum_{\substack{S \subseteq B(y) \\ \vee S = x}} (-1)^{|S|+1} \mu_Q(y).$$

*Proof.* For $y \in Q$, $f^{-1}(V_y)$ is an ascending subset of $P$ so it has suprema which coincide with those in $P$. Thus $f^{-1}(V_y)_{\hat{0}}$ is a lattice. We wish to compute $-\mu_{f^{-1}(V_y)}(x)$ for $x \in f^{-1}(V_y)$. By Rota's cross-cut theorem [32, Theorem 3], we know that for any finite lattice $L$ with $x \in L \backslash \{\hat{0}\}$,

$$\mu_L(\hat{0}, x) = \sum_{\substack{S \subseteq \text{atoms of } L \\ \vee S = x}} (-1)^{|S|}.$$

Therefore,

$$-\mu_{f^{-1}(V_y)}(x) = \sum_{\substack{S \subseteq B(y) \\ \vee S = x}} (-1)^{|S|+1},$$

since $B(y)$ is the set of atoms of the lattice $f^{-1}(V_y)_{\hat{0}}$. Q.E.D.

## 6. THE CRAPO COMPLEMENTATION THEOREM

In Crapo's original derivation [7] of the complementation theorem, he proves a theorem that, superficially at least, resembles Rota's Galois connection theorem. We see that this is no accident. Indeed we see that Crapo's theorem is the combinatorial analog of Theorem 4.1, just as Rota's theorem is the analog of the Leray spectral sequence.

We first state Crapo's theorem in a form equivalent to his original statement [7, Theorem 1] but which is more convenient for our purposes. We use the notation $P/S$, where $S \subseteq P$ is a descending subset of $P$, to denote the ordered set obtained by identifying (or collapsing) the elements of $S$.

THEOREM 6.1 (Crapo). *Let $P$ be a lower finite ordered set, and let $S \subseteq P$ be a descending subset. Then for any $x \in S$, $y \in P \backslash S$,*

$$\mu_P(x, y) = \sum_{z \in S} \mu_{[z,y]/S}(z, y)\, \mu_P(x, z),$$

*where $[z, y]/S$ is short for $[z, y]/(S \cap [z, y])$.*

*Proof.* We may clearly assume that $x$ is the minimum of $P$ while $y$ is the maximum, for we may replace $P$ by $[x, y]$ and $S$ by $[x, y] \cap S$ without changing the conclusion. We set $T = S \backslash \{x\}$, and let $B$: $Q \to T$ be given by $B(z) = \{t \mid t \leqslant z \text{ in } P\}$. Then $T_{\hat{0}} \cong S$ and $(Q +_B T)_{\hat{0}} \cong P$, as in Proposition 3.4.

We apply Theorem 4.1 to the order-preserving relation $B$ and the diagram $D = R[y]$ on $Q$. Then there is a spectral sequence

$$E_2^{p,q} = \tilde{H}^p(T, R^q B_* R[y]) \Rightarrow H^{n+1}(Q +_B T, R[y]).$$

We first compute the Euler characteristic of $E_2^{p,q}$:

$$\begin{aligned}
&\sum_{p,q} (-1)^{p+q} \operatorname{rank}_R \tilde{H}^p(T, R^q B_* R[y]) \\
&\quad = \sum_q (-1)^q \tilde{\chi}_T(R^q B_* R[y]) \\
&\quad = \sum_q (-1)^q \sum_{t \in T_{\hat{0}}} \tilde{\chi}_T((R^q B_* R[y])[t]) \\
&\quad = -\sum_{t \in S} \sum_q (-1)^q \operatorname{rank}_R(H^q(B^{-1}(V_t), R[y]))\, \mu_P(x, t) \\
&\quad = -\sum_{t \in S} \chi_{B^{-1}(V_y)}(R[y])\, \mu_P(x, t) \\
&\quad = \sum_{t \in S} \mu_{B^{-1}(V_y)}(\hat{0}, y)\, \mu_P(x, t) \\
&\quad = \sum_{t \in S} \mu_{[t,y]/S}(t, y)\, \mu_P(x, t).
\end{aligned}$$

Here we use the convention that $B^{-1}(V_{\hat{0}}) = Q$; see Section 2.6. It is easy to see that $B^{-1}(V_t)_{\hat{0}} \cong [t, y]/S$.

Next we compute the Euler characteristic of the abutment:

$$\sum_{n=-1}^{\infty} (-1)^n \operatorname{rank}_R H^{n+1}(Q +_B T, R[y]) = -\chi_{Q+_BT}(R[y])$$

$$= \mu_P(x, y). \qquad \text{Q.E.D.}$$

Now let $L$ be a finite lattice. Assume, to avoid trivialities, that $\hat{0} < \hat{1}$ and that $(\hat{0}, \hat{1}) \neq \varnothing$. Let $x \in (\hat{0}, \hat{1})$. A *complement* of $x$ is an element $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. We write $x \perp y$ in this case. Crapo [7, Theorem 2] showed that in this case

$$\mu_L(\hat{0}, \hat{1}) = \sum_{\substack{y \leqslant z \\ y, z \perp x}} \mu_L(\hat{0}, y)\, \mu_L(z, \hat{1}).$$

In particular, a noncomplemented lattice has $\mu_L(\hat{0}, \hat{1}) = 0$. We now show homological analogs of these results.

THEOREM 6.2. *Let $L$ be a finite lattice and $x \in (\hat{0}, \hat{1})$. Then there exist augmented diagrams $D^q$ supported on the complements of $x$ in $L$ and a spectral sequence*

$$E_2^{p,q} = \tilde{H}^p((\hat{0}, \hat{1}), D^q) \Rightarrow \tilde{H}^n((\hat{0}, \hat{1}), R).$$

*Proof.* Define $Q = \{y \in (\hat{0}, \hat{1}) \mid x \wedge y = \hat{0}\}$, $P = (\hat{0}, \hat{1}] \backslash Q$ and $A \colon P \to Q$ by $A(z) = \{y \in Q \mid y \leqslant z\}$. $Q$ is the set of nonzero lower semicomplements of $x$. Note that $Q$ is a descending subset of $(\hat{0}, \hat{1})$ and that we chose $A$ so that $L \simeq (P +_A Q)_{\hat{0}}$. We set $D^q = R^q A_* R[\hat{1}]$, and we use the spectral sequence in the proof of Theorem 6.1. $H^{n+1}((\hat{0}, \hat{1}], R[\hat{1}]) \simeq \tilde{H}^n((\hat{0}, \hat{1}), R)$ by [2, Lemma 3.1]. It remains to show that $D^q$ is supported on the complements of $x$.

First we let $t \in Q$. Then $(R^q A_* R[\hat{1}])_t = H^q(A^{-1}(V_t), R[\hat{1}])$. Now $A^{-1}(V_t) = P \cap V_t$. Let $y \in P \cap V_t$. Set $s = y \wedge (x \vee t)$. Then $s \wedge x = y \wedge (x \vee t) \wedge x = y \wedge x \neq \hat{0}$. Therefore $s \in P \cap V_t$ also. But $s \leqslant y$ and $s \leqslant x \vee t$. Since $y$ was arbitrary, we conclude that the supremum of the minimal elements of $P \cap V_t$ is below $x \vee t$.

It now follows from the cofinality result mentioned at the end of Section 4 that $x \vee t \neq \hat{1}$ implies $H^q(A^{-1}(V_t), R[\hat{1}]) = 0$, for we know by this result that $(P \backslash \{\hat{1}\}) \cap V_t$ and $J_{x \vee t} \cap V_t$ have the same reduced simplicial homology while $J_{x \vee t} \cap V_t$ has a maximum element.

Finally we consider the case $t = \hat{0}$, i.e., $(R^q A_*[\hat{1}])_{\hat{0}} = H^q(P, R[\hat{1}])$. This vanishes by precisely the same reasoning as above. Q.E.D.

COROLLARY 6.3. *Let $L$ be a finite noncomplemented lattice such that $(\hat{0}, \hat{1}) \neq \varnothing$. Then $(\hat{0}, \hat{1})$ has trivial reduced simplicial homology.*

If the complements of $x$ form an antichain, the diagrams $R^q A_* R[\hat{1}]$ are direct sums of skyscraper diagrams on the support. This does not necessarily mean that the spectral sequence degenerates. However if $L$ is Cohen–Macaulay (see [4]), we may conclude that it does. Indeed, in this case we have

$$H_{n-2}((\hat{0}, \hat{1}), \mathbb{Z}) \cong \bigoplus_{y \perp x} H_{r(y)-2}((\hat{0}, y), \mathbb{Z}) \otimes H_{n-r(y)-2}((y, \hat{1}), \mathbb{Z}).$$

See Baclawski [4] for details.

## Acknowledgments

The author would like to thank Heisuke Hironaka, Gian-Carlo Rota, and the referee for their many helpful comments and suggestions.

## References

1. M. Artin, "Grothendieck Topologies," Harvard Univ. Notes, 1962.
2. K. Baclawski, Whitney numbers of geometric lattices, *Advances in Math.* **16** (1975), 125–138.
3. K. Baclawski, "Homology and Combinatorics of Ordered Sets," Ph.D. Thesis, Harvard University, 1976.
4. K. Baclawski, Cohen–Macaulay ordered sets, in preparation.
5. G. Birkhoff, "Lattice Theory," 3rd ed., Amer. Math. Soc. Colloq. Publ., 25, Amer. Math. Soc., Providence, R.I., 1967.
6. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, N.J., 1956.
7. H. H. Crapo, The Möbius function of a lattice, *J. Combinatorial Theory* **1** (1966), 126–131.
8. H. H. Crapo, Möbius inversion in lattices, *Arch. Math. (Basel)* **19** (1968), 595–607.
9. "Möbius Algebras: Conference Proceedings" (H. H. Crapo and G. Roulet, eds.), Univ. of Waterloo, 1971.
10. R. Deheuvels, Homologie des ensembles ordonnés et des espaces topologiques, *Bull. Soc. Math. France* **90** (1962), 261–321.
11. J. C. Derdérian, Residuated mappings, *Pacific J. Math.* **20** (1967), 35–43.
12. J. C. Derdérian, Caractérisation des treillis de Brouwer et de Boole, *C. R. Acad. Sci. Paris Ser. A-B* **266** (1968), A441–442.
13. J. C. Derdérian, Généralisation d'un lemma de Fitting, *C. R. Acad. Sci. Paris Ser. A-B* **266** (1968), A1089–1090.
14. J. C. Derdérian, Galois connections and pair algebras, *Canad. J. Math.* **21** (1969), 498–501.
15. S. Eilenberg and N. Steenrod, "Foundations of Algebraic Topology," Princeton Univ. Press, Princeton, N.J., 1952.
16. C. J. Everett, Closure operators and Galois theory in lattices, *Trans. Amer. Math. Soc.* **55** (1944), 514–525.
17. F. Farmer, "Cellular Homology for Posets," preprint.
18. S. Fisk, Geometric coloring theory, *Advances in Math.* **24** (1977), 298–340.
19. J. Folkman, The homology groups of a lattice, *J. Math. Mech.* **15** (1966), 631–636.
20. R. Godement, "Topologie Algébrique et Théorie des Faisceaux," Hermann, Paris, 1958.

21. C. GREENE, On the Möbius algebra of a partially ordered set, *Advances in Math.* **10** (1973), 177–187.
22. H. B. GRIFFITHS, The homology groups of some ordered systems, *Acta Math.* **129** (1972), 195–235.
23. H. B. GRIFFITHS, An exact homology sequence induced by a Galois connection, *Proc. Lond. Math. Soc.* **32** (3) (1972), 101–116.
24. R. HARTSHORNE, Local cohomology, *in* "Lecture Notes in Mathematics," Vol. 41, Springer-Verlag, Berlin, 1967.
25. G. KEMPF, Local cohomology, in preparation.
26. B. MITCHELL, Rings with several objects, *Advances in Math.* **8** (1972), 1–161.
27. D. MUMFORD, "Introduction to Algebraic Geometry," (preliminary version of first three chapters), Harvard University.
28. O. ORE, Galois connexions, *Trans. Amer. Math. Soc.* **55** (1944), 493–513.
29. D. QUILLEN, "Higher Algebraic *K*-Theory, I," Lecture Notes in Mathematics, No. 341, Springer-Verlag, Berlin, 1973, pp. 85–147.
30. G. N. RANEY, Tight Galois connections and complete distributivity, *Trans. Amer. Math. Soc.* **97** (1960), 418–426.
31. G.-C. ROTA, On the combinatorics of the Euler characteristic, *Studies in Pure Math.* (1971), 221–233. (Rado Festschrift issue).
32. G.-C. ROTA, On the foundations of Combinatorial Theory. I: Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368.
33. Z. SHMUELY, The structure of Galois connections, *Pacific J. Math.* **54** (1974), 209–225.
34. E. SZPILRAJN, Sur l'extension de l'ordre partiel, *Fund. Math.* **16** (1930), 386–389.
35. A. WATERMAN, "General Valued Polarities," Thesis, Harvard University, 1971.